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Problem (principal stresses and principal directions of a stress tensor – an eigenvalue problem)
For given components of the stress tensor $\boldsymbol{\sigma}$ in the coordinate system $\{\boldsymbol{e}_i\}$ at a point P:

$$[\boldsymbol{\sigma}] = \frac{1}{5}\begin{bmatrix} 284 & 0 & 288 \\ 0 & 10 & 0 \\ 288 & 0 & 116 \end{bmatrix}_{\boldsymbol{e}_i} \text{MPa} \tag{1}$$

(a) determine principal stresses σ_i $(i = 1, 2, 3)$, and order them as $\sigma_1 \geq \sigma_2 \geq \sigma_3$

(b) determine principal directions $\boldsymbol{n}_i$ corresponding to stresses σ_i

(c) calculate the stress vector $\boldsymbol{f}^{(\boldsymbol{n})}$ at the point P on the plane π of normal vector $\boldsymbol{n}$,

$$\boldsymbol{n} = \frac{1}{\sqrt{61}}\,(3\boldsymbol{e}_1 + 4\boldsymbol{e}_2 + 6\boldsymbol{e}_3)$$

the length of normal $\boldsymbol{f}_N^{(\boldsymbol{n})}$ and tangential $\boldsymbol{f}_S^{(\boldsymbol{n})}$ components of $\boldsymbol{f}^{(\boldsymbol{n})}$, and $\alpha = \measuredangle(\boldsymbol{n}, \boldsymbol{f}^{(\boldsymbol{n})})$.

Solution

To (a) The eigenvalue problem for the stress tensor $\boldsymbol{\sigma}$ given in (1) is defined by the system

$$\begin{bmatrix} 56.80 - \sigma & 0 & 57.60 \\ 0 & 2 - \sigma & 0 \\ 57.60 & 0 & 23.20 - \sigma \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{2a}$$

which we supplement with the normalizing condition on the unknown vector $\boldsymbol{n}$, $\boldsymbol{n} \cdot \boldsymbol{n} = 1$,

$$n_1^2 + n_2^2 + n_3^2 = 1 \tag{2b}$$

The characteristic equation for $\boldsymbol{\sigma}$ can be obtained from $\det(\boldsymbol{\sigma} - \sigma\boldsymbol{I}) \equiv |\boldsymbol{\sigma} - \sigma\boldsymbol{I}| = 0$,

$$\begin{vmatrix} 56.80 - \sigma & 0 & 57.60 \\ 0 & 2 - \sigma & 0 \\ 57.60 & 0 & 23.20 - \sigma \end{vmatrix} = (2 - \sigma)[(56.80 - \sigma)(23.20 - \sigma) - 57.60^2] = 0 \tag{3}$$

or, by making use of invariants of $\boldsymbol{\sigma}$: $I_1 = 82$, $I_2 = -1840$, $I_3 = -4000$, which leads to

$$\sigma^3 - 82\,\sigma^2 - 1840\,\sigma + 4000 = 0 \tag{4}$$

The ordered solutions of (3) and (4) are: $\sigma_1 = 100$, $\sigma_2 = 2$, $\sigma_3 = -20$ MPa. Thus, we have

$$[\boldsymbol{\sigma}] = \frac{1}{5}\begin{bmatrix} 284 & 0 & 288 \\ 0 & 10 & 0 \\ 288 & 0 & 116 \end{bmatrix}_{\boldsymbol{e}_i} = \begin{bmatrix} 100 & & \\ & 2 & \\ & & -20 \end{bmatrix}_{\boldsymbol{n}_i} \text{MPa} \tag{5}$$

To (b) By substituting $\sigma = \sigma_i$ into (2a) and using (2b), we obtain the pairs $\{\sigma_i, \boldsymbol{n}_i\}$:

$$\begin{aligned} \sigma_1 &= 100\,\text{MPa} &\leftrightsquigarrow\quad \boldsymbol{n}_1 &= \frac{4}{5}\boldsymbol{e}_1 + 0\boldsymbol{e}_2 + \frac{3}{5}\boldsymbol{e}_3 \\ \sigma_2 &= 2\,\text{MPa} &\leftrightsquigarrow\quad \boldsymbol{n}_2 &= 0\boldsymbol{e}_1 + 1\boldsymbol{e}_2 + 0\boldsymbol{e}_3 \\ \sigma_3 &= -20\,\text{MPa} &\leftrightsquigarrow\quad \boldsymbol{n}_3 &= -\frac{3}{5}\boldsymbol{e}_1 + 0\boldsymbol{e}_2 + \frac{4}{5}\boldsymbol{e}_3 \end{aligned} \tag{6}$$

To (c) Direct calculation shows that

$$\boldsymbol{f}^{(\boldsymbol{n})} = \boldsymbol{\sigma}\boldsymbol{n} = 66.067\boldsymbol{e}_1 + 1.024\boldsymbol{e}_2 + 39.947\boldsymbol{e}_3$$

$$f^{(\boldsymbol{n})} = |\boldsymbol{f}^{(\boldsymbol{n})}| = 77.212\,\text{MPa}; \quad f_N^{(\boldsymbol{n})} = \boldsymbol{n} \cdot \boldsymbol{f}^{(\boldsymbol{n})} = 56.590\,\text{MPa}$$

$$f_S^{(\boldsymbol{n})} = \sqrt{(f^{(\boldsymbol{n})})^2 - (f_N^{(\boldsymbol{n})})^2} = 52.529\,\text{MPa}; \quad \cos\alpha = f_N^{(\boldsymbol{n})}/f^{(\boldsymbol{n})} = 0.7329; \quad \alpha = 42.87^\circ.$$